

# Regularization by noise in one-dimensional continuity equation.

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## Abstract

A linear stochastic continuity equation with non-regular coefficients is considered. We prove existence and uniqueness of strong solution, in the probabilistic sense to the Cauchy problem when the vector field has low regularity, in which the classical DiPerna-Lions-Ambrossio theory of uniqueness of distributional solutions does not apply. We solve partially the open problem that is the case when the vector-field has random dependence. Some motivations to study the above problem arise from the non-local conservation law. In addition, we prove a stability result for the solutions.

## 1 Introduction

Last decades the continuity equation has attracted a lot of scientific interest. The reason is that arises in a variety of domains such as biology, particle physics, population dynamics, crowd modeling, that can be modeled by the continuity/ transport equation,

$$\partial_t u(t, x) + \operatorname{div}(b(t, x)u(t, x)) = 0, \quad (1.1)$$

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where  $u$  is the physical quantity that evolves in time. Such quantities are for instance the vorticity of a fluid, or the density of a collection of particles advected by a velocity field which is highly irregular, in the sense that it has a derivative given by a distribution and a nonlinear dependence on the solution  $u$ .

When the coefficients are regular the unique solution is found by the method of characteristics. Recently research activity has been devoted to study continuity/transport equations with rough coefficients, showing a well-posedness result. . A complete theory of distributional solutions, including existence, uniqueness and stability properties, is provided in the seminal works of DiPerna and Lions [12] and Ambrosio [1].

The approach of DiPerna, Lions and Ambrosio relies on the theory of renormalized solutions. Roughly speaking, renormalized solutions are distributional solutions to which the chain rule applies in the sense that, for every suitable  $\beta$ ,  $\beta(u)$  solves the following continuity equation :

$$\partial_t \beta(u(t, x)) + \operatorname{div}(b(t, x) \cdot \beta(u(t, x))) = 0. \quad (1.2)$$

Whether distributional solutions are renormalized solutions depends on the regularity of  $b$ . In the paper by DiPerna and Lions was proved that when  $b$  has  $W^{1,1}$  spatial regularity (together with a condition of boundedness on the divergence) the commutator lemma between smoothing convolution and weak solution can be proved and, as a consequence, all  $L^\infty$ -weak solutions are renormalized. L. Ambrosio [1] generalized the theory to the case of only  $BV$  regularity for  $b$  instead of  $W^{1,1}$ . Another approach giving explicit compactness estimates has been introduced in [10], and further developed in [6, 19], see also the references therein. In the case of two-dimensional vector-field, we also refer to the work of F. Bouchut and L. Desvillettes [7] that treated the case of divergence free vector-field with continuous coefficient, and to [18] in which this result is extended to vector-field with  $L^2_{loc}$  coefficients with a condition of regularity on the direction of the vector-field. We refer the readers to two excellent summaries in [3] and [11]. For some recent developments see [8] and [29].

In contrast with its deterministic counterpart, the singular stochastic continuity/transport equation with multiplicative noise is well-posed. an ideal fluid in porous media. The addition of a stochastic noise is often used to account for numerical, empirical or physical uncertainties. The questions of regularizing effects and well-posedness by noise for (stochastic) par-

tial differential equations have attracted much interest in recent years. In [4, 5, 13, 14, 17, 25, 26, 27], well-posedness and regularization by linear multiplicative noise for continuity/transport equations, that is for

$$\partial_t u(t, x) + \operatorname{Div}\left(\left(b(t, x) + \frac{dB_t}{dt}\right) \cdot u(t, x)\right) = 0, \quad (1.3)$$

have been obtained. We refer to [26] for more details on the literature.

We report here the observation that a multiplicative noise as the one used in (1.3) is not enough to improve the regularity of solutions of the following stochastic conservation law

$$\partial_t u(t, x) + \partial_x u(t, x) \left(u(t, x) + \frac{dB_t}{dt}(\omega)\right) = 0.$$

Indeed, for this equation one can observe the appearance of shocks in finite time, just as for the deterministic conservation law, see [16]. For a different approach related to stochastic scalar conservation laws, we address the reader to [23] and [24].

The gap that the theory has, in order to work with the nonlinear systems where  $b$  depend on some quantity of  $u$ , is that in all cases  $b$  is deterministic vector field.

The purpose of the present paper is a contribution to the following general question: can one hope for an existence/uniqueness theory in the case where  $b$  is a stochastic process and it has low regularity. We present the first positive result. More precisely, we study the continuity equation

$$\begin{cases} \partial_t u(t, x) + \operatorname{Div}\left(\left(b(t, x, \omega) + \frac{dB_t}{dt}\right) \cdot u(t, x)\right) = 0, \\ u|_{t=0} = u_0. \end{cases} \quad (1.4)$$

Here,  $(t, x) \in [0, T] \times \mathbb{R}$ ,  $\omega \in \Omega$  is an element of the probability space  $(\Omega, \mathbb{P}, \mathcal{F})$ ,  $b : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is a given vector field,  $B_t$  is a standard Brownian motion. The stochastic integration is to be understood in the Stratonovich sense.

The novelty of our results is to show existence and uniqueness of the solutions for one-dimensional stochastic continuity equation (1.4) when the vector field  $b$  has random dependence and when it is bounded and integrable without assumptions on the divergence.

Here, of course, the first difficulty is in the existence part, since standard approximation schemes in general do not provide existence without assumptions on the divergence. The result is based on the regularization effect of the Brownian perturbation on the flow of the characteristics equation

$$X_t = x + \int_0^t b(s, X_s(x), \omega) ds + B_t. \quad (1.5)$$

For the key estimates on the spatial weak derivatives of solutions of the SDE (1.5) we use stochastic calculus technique.

Let us describe in few words the strategy of the uniqueness proof. It is based on the fact that one primitive  $V$  is regular and verifies the transport equation

$$\partial_t V(t, x) + (b(t, x, \omega) + \frac{dB_t}{dt}) \cdot \nabla V(t, x) = 0. \quad (1.6)$$

Then using a modified version of the commutator Lemma and the characteristics systems associated to the SPDE (1.6) we shall show that  $V = 0$  with initial condition equal to zero, which implies that  $u = 0$ .

Other pint in this paper is to show a stability result for the Cauchy problem (1.4).

## 1.1 Hypothesis.

In this paper we assume the following hypothesis:

**Hypothesis 1.1.** *The vector field  $b$  satisfies*

$$b \in L^\infty(\Omega \times [0, T], L^1(\mathbb{R})), \quad (1.7)$$

$$b \in L^\infty(\Omega \times [0, T] \times \mathbb{R}), \quad (1.8)$$

$$b(t, x, \omega) = \int_0^t f(s, x, \omega) ds + \int_0^t g(s, x, \omega) dB_s, \quad (1.9)$$

where

$$f \in L^\infty(\Omega, L^1([0, T] \times \mathbb{R}))$$

and

$$g \in L^\infty(\Omega \times \mathbb{R}, L^1([0, T])) \cap L^\infty(\Omega, L^2([0, T], L^1(\mathbb{R}))).$$

Moreover, the initial condition verifies

$$u_0 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}). \quad (1.10)$$

## 1.2 Notations

First, through of this paper, we fix a stochastic basis with a  $d$ -dimensional Brownian motion  $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0, T]\}, \mathbb{P}, (B_t))$ . Then, we recall to help the intuition, the following definitions

$$\text{It\^o: } \int_0^t X_s dB_s = \lim_{n \rightarrow \infty} \sum_{t_i \in \pi_n, t_i \leq t} X_{t_i} (B_{t_{i+1} \wedge t} - B_{t_i}),$$

$$\text{Stratonovich: } \int_0^t X_s \circ dB_s = \lim_{n \rightarrow \infty} \sum_{t_i \in \pi_n, t_i \leq t} \frac{(X_{t_i \wedge t} + X_{t_i})}{2} (B_{t_{i+1} \wedge t} - B_{t_i}),$$

$$\text{Covariation: } [X, Y]_t = \lim_{n \rightarrow \infty} \sum_{t_i \in \pi_n, t_i \leq t} (X_{t_i \wedge t} - X_{t_i})(Y_{t_{i+1} \wedge t} - Y_{t_i}),$$

where  $\pi_n$  is a sequence of finite partitions of  $[0, T]$  with size  $|\pi_n| \rightarrow 0$  and elements  $0 = t_0 < t_1 < \dots$ . The limits are in probability, uniformly in time on compact intervals. Details about these facts can be found in Kunita [21]. Also we address from that book, It\^o's formula, the chain rule for the stochastic integral, for any continuous  $d$ -dimensional semimartingale  $X = (X_1, X_2, \dots, X_d)$ , and twice continuously differentiable and real valued function  $f$  on  $\mathbb{R}^d$ .

## 1.3 Motivations.

Apart from the theoretical importance of such an extension, the main motivation comes from the study of many nonlinear partial differential equations of the mathematical physics. In various physical models of the mechanics of fluids it is essential to deal with densities or with velocity fields which are not smooth and this corresponds to effective real world situations. This is our motivation to study the effect of the noise in transport/continuity equation

when the drift  $b$  is a stochastic process. In particular, we are interested in to show uniqueness of weak solutions for no-local conservation law, that is for the equation

$$\partial_t u(t, x) + \text{Div}\left((F(t, x, (K * u)(x)) + \frac{dB_t}{dt}) \cdot u(t, x)\right) = 0,$$

where  $K$  is a regular kernel. Our conjecture is the uniqueness for type of the conservation law in opposition to the deterministic theory where the solutions are unique under entropy condition.

## 2 Estimation for the flow.

To begin, let us consider the stochastic differential equation in  $\mathbb{R}$ , that is to say, given  $s \in [0, T]$  and  $x \in \mathbb{R}$ , we consider

$$X_{s,t}(x) = x + \int_s^t b(u, X_{s,u}(x), \omega) du + B_t - B_s, \quad (2.11)$$

where  $X_{s,t}(x) = X(s, t, x)$  (also  $X_t(x) = X(0, t, x)$ ). In particular, for  $m \in \mathbb{N}$  and  $0 < \alpha < 1$ , we assume

$$b \in L^1([0, T]; (C^{m,\alpha}(\mathbb{R}))). \quad (2.12)$$

It is well known that, under the above regularity of the drift vector field  $b$ , the stochastic flow  $X_{s,t}$  is a  $C^m$  diffeomorphism (see for example [9, 20]). Moreover, the inverse  $Y_{s,t} := X_{s,t}^{-1}$  satisfies the following backward stochastic differential equations,

$$Y_{s,t} = y - \int_s^t b(u, Y_{u,t}) du - (B_t - B_s), \quad (2.13)$$

for  $0 \leq s \leq t$ .

SDEs with discontinuous coefficients and driven by Brownian motion have been an important area of study in stochastic analysis and has been a very active topic of research in the last years. The method of stochastic characteristics may be employed to prove uniqueness of solutions of the stochastic transport/continuity equation under weak regularity hypotheses on the drift coefficient. In this way we have the next estimation.

**Lemma 2.1.** Assume  $b \in C_b^\infty(\mathbb{R})$  and that satisfies the hypothesis 1.1. Then for  $T > 0$  there exist a constant  $C$  such that

$$\mathbb{E} \left[ \left| \frac{d}{dx} X_{s,t}(x) \right|^{-1} \right] \leq C. \quad (2.14)$$

where  $C$  on  $T$ ,  $\|b\|_{L^\infty([0,T] \times \Omega, L^1(\mathbb{R}))}$ ,  $\|b\|_{L^\infty((\Omega \times \mathbb{R}), L^2([0,T])}^2$ ,  $\|f\|_{L^\infty(\Omega, L^1([0,T] \times \mathbb{R}))}$ ,  $\|g\|_{L^\infty(\Omega \times \mathbb{R}, L^1([0,T]))}$  and  $\|g\|_{L^\infty(\Omega \times \mathbb{R}, L^2(\mathbb{R}, L^1([0,T])))}$ .

*Proof.* We consider the SDE associated to the vector field  $b$  :

$$dX_t = b(t, X_t, \omega) dt + dB_t, \quad X_s = x.$$

We note that  $\partial_x X_{s,t}$  satisfies

$$\partial_x X_{s,t} = \exp \left\{ \int_s^t b'(X_{s,u}) du \right\}.$$

We denoted

$$\begin{aligned} \tilde{b}(t, z, \omega) &= \int_\infty^z b(t, y, \omega) dy, \\ \tilde{g}(t, z, \omega) &= \int_\infty^z g(t, y, \omega) dy, \\ \tilde{f}(t, z, \omega) &= \int_\infty^z f(t, y, \omega) dy. \end{aligned}$$

Applying the Itô-Wentzell-Kunita formula to  $\tilde{b}$ , see Theorem 8.3 of [21], we have

$$\begin{aligned} \tilde{b}(t, X_{s,t}, \omega) &= \tilde{b}(s, x, \omega) + \int_s^t \tilde{f}(u, X_{s,u}, \omega) du + \int_s^t \tilde{g}(u, X_{s,u}, \omega) dB_u + \int_s^t b^2(u, X_{s,u}, \omega) du \\ &\quad + \int_s^t g(u, X_{s,u}, \omega) du + \int_s^t b(u, X_{s,u}, \omega) dB_u + \frac{1}{2} \int_s^t b'(u, X_{s,u}, \omega) du. \end{aligned}$$

We set

$$\mathcal{E} \left( \int_s^t b(u, X_{s,u}, \omega) dB_u \right) = \exp \left\{ \int_s^t b(u, X_{s,u}, \omega) dB_u - \frac{1}{2} \int_s^t b^2(u, X_{s,u}, \omega) du \right\},$$

and

$$\mathcal{E}\left(\int_s^t \tilde{g}(u, X_{s,u}, \omega) dB_u\right) = \exp\left\{\int_s^t \tilde{g}(u, X_{s,u}, \omega) dB_u - \frac{1}{2} \int_s^t \tilde{g}^2(u, X_{s,u}, \omega) du\right\}.$$

Now, we observe

$$\|\tilde{b}\|_\infty \leq \|b\|_{L^\infty([0,T] \times \Omega, L^1(\mathbb{R}))}, \quad (2.15)$$

$$\left\|\int_s^t b^2(X_{s,u}) du\right\|_\infty \leq C\|b\|_{L^\infty((\Omega \times \mathbb{R}), L^2([0,T])})}^2, \quad (2.16)$$

$$\left\|\int_s^t \tilde{f}(u, X_{s,u}, \omega) du\right\|_\infty \leq C\|f\|_{L^\infty(\Omega, L^1([0,T] \times \mathbb{R}))}, \quad (2.17)$$

$$\left\|\int_s^t g(u, X_{s,u}, \omega) du\right\|_\infty \leq C\|g\|_{L^\infty(\Omega \times \mathbb{R}, L^1([0,T]))}, \quad (2.18)$$

$$\left\|\int_s^t |\tilde{g}(s, X_{s,u}, \omega)|^2 du\right\|_\infty \leq C\|g\|_{L^\infty(\Omega, L^2(\mathbb{R}, L^1([0,T])))}. \quad (2.19)$$

Thus we get that

$$\begin{aligned} \mathbb{E}\left[\left|\frac{dX_{s,t}}{dx}(x)\right|^{-1}\right] &= \mathbb{E}\left[\exp\left\{\frac{1}{2}\left[-\tilde{b}(t, X_{s,t}, \omega) + \tilde{b}(s, x, \omega) + \int_s^t b^2(u, X_{s,u}, \omega) du\right.\right.\right. \\ &\quad \left.\left.+ \int_s^t b(u, X_{s,u}, \omega) dB_u + \int_s^t \tilde{f}(u, X_{s,u}, \omega) du +\right.\right. \\ &\quad \left.\left.+ \int_s^t g(u, X_{s,u}, \omega) du + \int_s^t \tilde{g}(u, X_{s,u}, \omega) dB_u\right]\right\} \\ &\leq C\mathbb{E}\left[\mathcal{E}\left(\int_s^t b(u, X_{s,u}, \omega) dB_u\right)\right] \times \mathbb{E}\left[\mathcal{E}\left(\int_s^t \tilde{g}(u, X_{s,u}, \omega) dB_u\right)\right] \end{aligned} \quad (2.20)$$

where we used the Hölder inequality.

Finally we observe that the processes  $\mathcal{E}\left(\int_0^t b(u, X_{s,u}, \omega) dB_s\right)$ ,  $\mathcal{E}\left(\int_0^t \tilde{g}(u, X_{s,u}, \omega) dB_s\right)$  are martingales with expectation equal to one. From this we conclude our lemma.  $\square$



**Remark 2.2.** *The same results is valid for the backward flow  $Y_{s,t}$  since it is solution of the same SDE driven by the drifts  $-b$ .*

### 3 $L^2$ - Solutions.

#### 3.1 Definition of solutions

**Definition 3.1.** *A stochastic process  $u \in L^\infty([0, T], L^2(\Omega \times \mathbb{R})) \cap L^1(\Omega \times [0, T] \times \mathbb{R})$  is called a  $L^2$ - weak solution of the Cauchy problem (1.4) when: For any  $\varphi \in C_0^\infty(\mathbb{R})$ , the real valued process  $\int u(t, x)\varphi(x)dx$  has a continuous modification which is an  $\mathcal{F}_t$ -semimartingale, and for all  $t \in [0, T]$ , we have  $\mathbb{P}$ -almost surely*

$$\begin{aligned} \int_{\mathbb{R}} u(t, x)\varphi(x)dx &= \int_{\mathbb{R}} u_0(x)\varphi(x) dx + \int_0^t \int_{\mathbb{R}} u(s, x) b(s, x, \omega) \partial_x \varphi(x) dx ds \\ &\quad + \int_0^t \int_{\mathbb{R}} u(s, x) \partial_x \varphi(x) dx \circ dB_s. \end{aligned} \tag{3.21}$$

**Remark 3.2.** *Using the same idea as in Lemma 13 [17], one can write the problem (1.4) in Itô form as follows, a stochastic process  $u \in L^\infty([0, T], L^2(\Omega \times \mathbb{R})) \cap L^1(\Omega \times [0, T] \times \mathbb{R})$  is a  $L^2$ - weak solution of the SPDE (1.4) iff for every test function  $\varphi \in C_0^\infty(\mathbb{R})$ , the process  $\int u(t, x)\varphi(x)dx$  has a continuous modification which is a  $\mathcal{F}_t$ -semimartingale and satisfies the following Itô's formulation*

$$\begin{aligned} \int_{\mathbb{R}} u(t, x)\varphi(x)dx &= \int_{\mathbb{R}} u_0(x)\varphi(x) dx + \int_0^t \int_{\mathbb{R}} u(s, x) b(s, x, \omega) \partial_x \varphi(x) dx ds \\ &\quad + \int_0^t \int_{\mathbb{R}} u(s, x) \partial_x \varphi(x) dx dB_s + \frac{1}{2} \int_0^t \int_{\mathbb{R}} u(s, x) \partial_x^2 \varphi(x) dx ds. \end{aligned}$$

#### 3.2 Existence.

The goal of this section is to prove general existence result for stochastic continuity equation without assumptions on the divergence.

**Lemma 3.3.** *Assume that hypothesis 1.1 holds. Then there exist  $L^2$ -weak solutions of the Cauchy problem (1.4).*

*Proof. Step 1: Regularization.*

Let  $\{\rho_\varepsilon\}_\varepsilon$  be a family of standard symmetric mollifiers and  $\eta$  a nonnegative smooth cut-off function supported on the ball of radius 2 and such that  $\eta = 1$  on the ball of radius 1. Now, for every  $\varepsilon > 0$ , we introduce the rescaled functions  $\eta_\varepsilon(\cdot) = \eta(\varepsilon \cdot)$ . Thus, we define the family of regularized coefficients given by

$$b^\varepsilon(x) = \eta_\varepsilon(x)(b * \rho_\varepsilon(x))$$

and

$$u_0^\varepsilon(x) = \eta_\varepsilon(x)(u_0 * \rho_\varepsilon(x)).$$

Clearly we observe that, for every  $\varepsilon > 0$ , any element  $b^\varepsilon, u_0^\varepsilon$  are smooth (in space) and have compactly supported with bounded derivatives of all orders. We consider the regularized version of stochastic continuity equation given by :

$$\begin{cases} du^\varepsilon(t, x) + \text{Div}\left(u^\varepsilon(t, x) \cdot (b^\varepsilon(t, x, \omega)dt + \circ dB_t)\right) = 0, \\ u^\varepsilon|_{t=0} = u_0^\varepsilon. \end{cases} \quad (3.22)$$

Following the classical theory of H. Kunita [20, Theorem 6.1.9] we obtain that

$$u^\varepsilon(t, x) = u_0^\varepsilon(\psi_t^\varepsilon(x))J\psi_t^\varepsilon(x)$$

is the unique solution to the regularized equation (3.22), where  $\phi_t^\varepsilon$  is the flow associated to the following stochastic differential equation (SDE):

$$dX_t = b^\varepsilon(X_t)dt + dB_t, \quad X_0 = x,$$

and  $\psi_t^\varepsilon$  is the inverse of  $\phi_t^\varepsilon$ .

*Step 2: Boundedness.* Making the change of variables  $y = \psi_t^\varepsilon(x) = (\phi_t^\varepsilon(x))^{-1}$  we have that

$$\int_{\mathbb{R}} \mathbb{E}[|u^\varepsilon(t, x)|^2] dx = \int_{\Omega} \int_{\mathbb{R}} |u_0^\varepsilon(y)|^2 (J\phi_t^\varepsilon(x))^{-1} dx.$$

Now, by Lemma 2.1 we obtain

$$\int_{\mathbb{R}} \mathbb{E}[|u^\varepsilon(t, x)|^2] dx \leq C \int_{\mathbb{R}} |u_0^\varepsilon(y)|^2 dx. \quad (3.23)$$

Therefore, the sequence  $\{u^\varepsilon\}_{\varepsilon>0}$  is bounded in  $u \in L^2(\Omega \times [0, T] \times \mathbb{R}) \cap L^\infty([0, T], L^2(\Omega \times \mathbb{R}))$ . Then there exists a convergent subsequence, which we denote also by  $u^\varepsilon$ , such that converge weakly in  $L^2(\Omega \times [0, T] \times \mathbb{R})$  and weak-star in  $L^\infty([0, T], L^2(\Omega \times \mathbb{R}))$  to some process  $u \in L^2(\Omega \times [0, T] \times \mathbb{R}) \cap L^\infty([0, T], L^2(\Omega \times \mathbb{R}))$ . Since this subsequence is bounded in  $L^1(\Omega \times [0, T] \times \mathbb{R})$  we follows that  $u^\varepsilon$  converge to the measure  $\mu$  and  $\mu = u$ .

*Step 3: Passing to the Limit.* Now, if  $u^\varepsilon$  is a solution of (3.22), it is also a weak solution, that is, for any test function  $\varphi \in C_0^\infty(\mathbb{R})$ ,  $u^\varepsilon$  verifies (written in the Itô form):

$$\begin{aligned} \int_{\mathbb{R}} u^\varepsilon(t, x) \varphi(x) dx &= \int_{\mathbb{R}} u_0^\varepsilon(x) \varphi(x) dx + \int_0^t \int_{\mathbb{R}} u^\varepsilon(s, x) b^\varepsilon(s, x, \omega) \partial_x \varphi(x) dx ds \\ &\quad + \int_0^t \int_{\mathbb{R}} u^\varepsilon(s, x) \partial_x \varphi(x) dx dB_s + \frac{1}{2} \int_0^t \int_{\mathbb{R}} u^\varepsilon(s, x) \partial_x^2 \varphi(x) dx ds. \end{aligned}$$

Then, for prove existence of the equation (1.4) is enough to pass to the limit in the above equation along the convergent subsequence found. This is made through of the same arguments of [17, theorem 15].

□

### 3.3 Uniqueness.

In this section, we shall present a uniqueness theorem for the SPDE (1.4). The proof is base on to apply the characteristic method and the commutator Lemma to a primitive of the solution. We pointed that similar arguments was used in previous wok [26].

**Theorem 3.4.** *Under the conditions of hypothesis 1.1, uniqueness holds for  $L^2$ - weak solutions of the Cauchy problem (1.4) in the following sense: if  $u, v$  are  $L^2$ - weak solutions with the same initial data  $u_0 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ , then  $u = v$  almost everywhere in  $\Omega \times [0, T] \times \mathbb{R}$ .*

*Proof. Step 0: Set of solutions.* Remark that the set of  $L^2$ - weak solutions is a linear subspace of  $L^2(\Omega \times [0, T] \times \mathbb{R})$ , because the stochastic continuity equation is linear, and the regularity conditions is a linear constraint. Therefore, it is enough to show that a  $L^2$ - weak solution  $u$  with initial condition  $u_0 = 0$  vanishes identically.

*Step 1: Primitive of the solution.* We set

$$V(t, x) = \int_{-\infty}^x u(t, y) dy.$$

We observe that  $\partial_x V(t, x) = u(t, x)$  belong to  $L^2(\Omega \times [0, T] \times \mathbb{R})$ . Now, we consider a nonnegative smooth cut-off function  $\eta$  supported on the ball of radius 2 and such that  $\eta = 1$  on the ball of radius 1. For any  $R > 0$ , we introduce the rescaled functions  $\eta_R(\cdot) = \eta(\frac{\cdot}{R})$ .

For all test functions  $\varphi \in C_0^\infty(\mathbb{R})$  we obtain

$$\int_{\mathbb{R}} V(t, x) \varphi(x) \eta_R(x) dx = - \int_{\mathbb{R}} u(t, x) \theta(x) \eta_R(x) dx - \int_{\mathbb{R}} V(t, x) \theta(x) \partial_x \eta_R(x) dx,$$

where  $\theta(x) = \int_{-\infty}^x \varphi(y) dy$ . By definition of  $L^2$ -solutions, taking as test function  $\theta(x) \eta_R(x)$  we get

$$\begin{aligned} \int_{\mathbb{R}} V(t, x) \eta_R(x) \varphi(x) dx &= - \int_0^t \int_{\mathbb{R}} \partial_x V(s, x) b(s, x, \omega) \eta_R(x) \varphi(x) dx ds \\ &\quad - \int_0^t \int_{\mathbb{R}} \partial_x V(s, x) \eta_R(x) \varphi(x) dx \odot dB_s - \int_0^t \int_{\mathbb{R}} \partial_x V(s, x) b(s, x, \omega) \partial_x \eta_R(x) \theta(x) dx ds \\ &\quad - \int_0^t \int_{\mathbb{R}} \partial_x V(s, x) \partial_x \eta_R(x) \theta(x) dx \odot dB_s - \int_{\mathbb{R}} V(t, x) \theta(x) \partial_x \eta_R(x) dx. \end{aligned} \tag{3.24}$$

We observe that

$$\int_0^t \int_{\mathbb{R}} \partial_x V(s, x) \partial_x \eta_R(x) \theta(x) dx \odot dB_s \rightarrow 0,$$

$$\int_{\mathbb{R}} V(t, x) \theta(x) \partial_x \eta_R(x) dx \rightarrow 0,$$

as  $R \rightarrow \infty$ . Passing to the limit in equation (3.24) we have that

$$\begin{aligned} &\int_{\mathbb{R}} V(t, x) \varphi(x) dx \\ &= - \int_0^t \int_{\mathbb{R}} \partial_x V(s, x) b(s, x, \omega) \varphi(x) dx ds - \int_0^t \int_{\mathbb{R}} \partial_x V(s, x) \varphi(x) dx \odot dB_s. \end{aligned}$$

*Step 2: Smoothing.* Let  $\{\rho_\varepsilon(x)\}_\varepsilon$  be a family of standard symmetric mollifiers. For any  $\varepsilon > 0$  and  $x \in \mathbb{R}^d$  we use  $\rho_\varepsilon(x - \cdot)$  as test function, then we deduce

$$\begin{aligned} \int_{\mathbb{R}} V(t, y) \rho_\varepsilon(x - y) dy &= - \int_0^t \int_{\mathbb{R}} (b(s, y, \omega) \partial_y V(s, y)) \rho_\varepsilon(x - y) dy ds \\ &\quad - \int_0^t \int_{\mathbb{R}} \partial_y V(s, y) \rho_\varepsilon(x - y) dy \circ dB_s \end{aligned}$$

We denote  $V_\varepsilon(t, x) = (V * \rho_\varepsilon)(x)$ ,  $b_\varepsilon(t, x, \omega) = (b * \rho_\varepsilon)(x)$  and  $(bV)_\varepsilon(t, x) = (b \cdot V * \rho_\varepsilon)(x)$ . Thus we have

$$\begin{aligned} V_\varepsilon(t, x) &+ \int_0^t b_\varepsilon(s, x, \omega) \partial_x V_\varepsilon(s, x) ds + \int_0^t \partial_x V_\varepsilon(s, x) \circ dB_s \\ &= \int_0^t (\mathcal{R}_\varepsilon(V, b))(x, s) ds, \end{aligned}$$

where we denote  $\mathcal{R}_\varepsilon(V, b) = b_\varepsilon \partial_x V_\varepsilon - (b \partial_x V)_\varepsilon$ .

*Step 3: Method of Characteristics.* Applying the Itô-Wentzell-Kunita formula to  $V_\varepsilon(t, X_t^\varepsilon)$ , see Theorem 8.3 of [21], we have

$$V_\varepsilon(t, X_t^\varepsilon) = \int_0^t (\mathcal{R}_\varepsilon(V, b))(X_s^\varepsilon, s) ds.$$

Then, considering that  $X_t^\varepsilon = X_{0,t}^\varepsilon$  and  $Y_t^\varepsilon = Y_{0,t}^\varepsilon = (X_{0,t}^\varepsilon)^{-1}$  we deduce that

$$V_\varepsilon(t, x) = \int_0^t (\mathcal{R}_\varepsilon(V, b))(X_{0,s}^\varepsilon(Y_{0,t}^\varepsilon), s) ds = \int_0^t (\mathcal{R}_\varepsilon(V, b))(Y_{s,t}^\varepsilon, s) ds.$$

Multiplying by the test functions  $\varphi$  and integrating in  $\mathbb{R}$  we get

$$\int_{\mathbb{R}} V_\varepsilon(t, x) \varphi(x) dx = \int_0^t \int_{\mathbb{R}} (\mathcal{R}_\varepsilon(V, b))(Y_{s,t}^\varepsilon, s) \varphi(x) dx ds. \quad (3.25)$$

Finally we observe that

$$\int_0^t \int (\mathcal{R}_\epsilon(V, b))(Y_{s,t}^\epsilon, s) \varphi(x) dx ds = \int_0^t \int (\mathcal{R}_\epsilon(V, b))(x, s) JX_{s,t}^\epsilon \varphi(X_{s,t}^\epsilon) dx ds. \quad (3.26)$$

*Step 4: Convergence of the commutator.* Now, we observe that  $\mathcal{R}_\epsilon(V, b)$  converge to zero in  $L^2([0, T] \times \Omega \times \mathbb{R})$ . In fact, we get that

$$(b \partial_x V)_\epsilon \rightarrow b \partial_x V \text{ in } L^2([0, T] \times \Omega \times \mathbb{R}).$$

Moreover, we have

$$b_\epsilon \rightarrow b \text{ in } L^1([0, T] \times \Omega \times \mathbb{R})$$

and

$$\partial_x V_\epsilon \rightarrow \partial_x V \text{ in } L^2([0, T] \times \Omega \times \mathbb{R}).$$

Then by the dominated convergence theorem we obtain

$$b_\epsilon \partial_x V_\epsilon \rightarrow b \partial_x V \text{ in } L^2([0, T] \times \Omega \times \mathbb{R}).$$

*Step 5: Conclusion.* From step 3 we have

$$\int V_\epsilon(t, x) \varphi(x) dx = \int_0^t \int (\mathcal{R}_\epsilon(V, b))(x, s) JX_{s,t}^\epsilon \varphi(X_{s,t}^\epsilon) dx ds. \quad (3.27)$$

By Hölder's inequality we obtain

$$\begin{aligned} & \mathbb{E} \left| \int_0^t \int (\mathcal{R}_\epsilon(V, b))(x, s) JX_{s,t}^\epsilon \varphi(X_{s,t}^\epsilon) dx ds \right| \\ & \leq \left( \mathbb{E} \int_0^t \int |(\mathcal{R}_\epsilon(V, b))(x, s)|^2 dx ds \right)^{\frac{1}{2}} \left( \mathbb{E} \int_0^t \int |JX_{s,t}^\epsilon \varphi(X_{s,t}^\epsilon)|^2 dx ds \right)^{\frac{1}{2}} \end{aligned}$$

From step 4 we follow

$$\left( \mathbb{E} \int_0^t \int |(\mathcal{R}_\epsilon(V, b))(x, s)|^2 dx ds \right)^{\frac{1}{2}} \rightarrow 0.$$

From lemma 2.1 we deduce

$$\begin{aligned} \left( \mathbb{E} \int_0^t \int |JX_{s,t}^\epsilon \varphi(X_{s,t}^\epsilon)|^2 dx ds \right)^{\frac{1}{2}} &= \left( \mathbb{E} \int_0^t \int |JY_{s,t}^\epsilon|^{-1} |\varphi(x)|^2 dx ds \right)^{\frac{1}{2}} \\ &\leq C \left( \int |\varphi(x)|^2 dx \right)^{\frac{1}{2}}, \end{aligned}$$

Passing to the limit in equation (3.27) we deduced that  $V = 0$ . Then we conclude that  $u = 0$ . □

### 3.4 Stability

To end up the well-posedness for the continuity equation (1.4), it remains to show the stability property for the solution with respect to the initial datum. We use the same ideas that in the uniqueness proof.

**Theorem 3.5.** *Assume hypothesis 1.1. Let  $\{u_0^n\}$  be any sequence, with  $u_0^n \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$  ( $n \geq 1$ ), converging strongly to  $u_0 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ . Let  $u(t, x)$ ,  $u^n(t, x)$  be the unique weak  $L^2$ -solution of the Cauchy problem (1.4), for respectively the initial data  $u_0$  and  $u_0^n$ . Then, for all  $t \in [0, T]$ , and for each function  $\varphi \in C_c^\infty(\mathbb{R}^d)$   $\mathbb{P}$ - a.s.*

$$\int_{\mathbb{R}^d} u^n(t, x) \varphi(x) dx \text{ converges to } \int_{\mathbb{R}^d} u(t, x) \varphi(x) dx \quad \mathbb{P}\text{- a.s..}$$

*Proof. Step 1: Primitives.* We set

$$V(t, x) = \int_{-\infty}^x u(t, y) dy,$$

and

$$V^n(t, x) = \int_{-\infty}^x u^n(t, y) dy.$$

Then we have

$$\int_{\mathbb{R}} V(t, x) \varphi(x) dx$$

$$= \int u_0(x) \varphi(x) dx - \int_0^t \int_{\mathbb{R}} \partial_x V(s, x) b(s, x, \omega) \varphi(x) dx ds - \int_0^t \int_{\mathbb{R}} \partial_x V(s, x) \varphi(x) dx \circ dB_s,$$

and

$$\int_{\mathbb{R}} V^n(t, x) \varphi(x) dx$$

$$= \int u_0^n(x) \varphi(x) dx - \int_0^t \int_{\mathbb{R}} \partial_x V^n(s, x) b(s, x, \omega) \varphi(x) dx ds - \int_0^t \int_{\mathbb{R}} \partial_x V^n(s, x) \varphi(x) dx \circ dB_s.$$

We set  $W^n = V(t, x) - V^n(t, x)$  and  $W_0^n = u_0(x) - u_0^n(x)$ . Thus we obtain que  $W^n$  verifies

$$\int_{\mathbb{R}} W^n(t, x) \varphi(x) dx$$

$$= \int W_0^n(x) \varphi(x) dx - \int_0^t \int_{\mathbb{R}} \partial_x W^n(s, x) b(s, x, \omega) \varphi(x) dx ds - \int_0^t \int_{\mathbb{R}} \partial_x W^n(s, x) \varphi(x) dx \circ dB_s.$$

*Step 2: Regularization.* We denote  $W_\varepsilon^n(t, x) = (W^n * \rho_\varepsilon)(x)$ ,  $b_\varepsilon(t, x, \omega) = (b * \rho_\varepsilon)(x)$  and  $(bW^n)_\varepsilon(t, x) = (b \cdot V * \rho_\varepsilon)(x)$ . Thus we have

$$W_\varepsilon^n(t, x) - W_0^{n, \varepsilon}(x) + \int_0^t b_\varepsilon(s, x, \omega) \partial_x W_\varepsilon^n(s, x) ds + \int_0^t \partial_x W_\varepsilon^n(s, x) \circ dB_s =$$

$$\int_0^t (\mathcal{R}_\varepsilon(W^n, b))(x, s) ds,$$

where we denote  $\mathcal{R}_\varepsilon(W^n, b) = b_\varepsilon \partial_x W_\varepsilon^n - (b \partial_x W^n)_\varepsilon$ .

*Step 3: Method of Characteristic.* Applying the Itô-Wentzell-Kunita formula to  $W_\varepsilon^n(t, X_t^\varepsilon)$ , see Theorem 8.3 of [21], we have

$$W_\varepsilon^n(t, X_t^\varepsilon) = W_0^{n, \varepsilon}(x) + \int_0^t (\mathcal{R}_\varepsilon(W^n, b))(X_s^\varepsilon, s) ds.$$

Then we obtain



$$W_\varepsilon^n(t, x) = W_0^{n,\varepsilon}(Y_t^\varepsilon) + \int_0^t (\mathcal{R}_\varepsilon(W^n, b))(Y_{s,t}^\varepsilon, s) ds.$$

Multiplying by the test functions  $\varphi$  and integrating in  $\mathbb{R}$  we get

$$\int W_\varepsilon^n(t, x) \varphi(x) dx = \int W_0^{n,\varepsilon}(Y_t^\varepsilon)(x) dx + \int_0^t \int (\mathcal{R}_\varepsilon(W^n, b))(Y_{s,t}^\varepsilon, s) \varphi(x) dx ds. \quad (3.28)$$

*Step 4: Conclusion.* Arguing as in step 5 of the theorem 3.4 we get

$$\int_0^t \int (\mathcal{R}_\varepsilon(W^n, b))(Y_{s,t}^\varepsilon, s) \varphi(x) dx ds \rightarrow 0$$

when  $\varepsilon$  converge to zero. Making change of variables we have

$$\int W_0^n(Y_t^\varepsilon)(x) \varphi(x) dx = \int W_0^n(x) JX_t^\varepsilon \varphi(X_t^\varepsilon) dx.$$

Thus we get

$$\begin{aligned} \mathbb{E} \left| \int W_0^n(x) JX_t^\varepsilon \varphi(X_t^\varepsilon) dx \right|^2 &\leq \int |W_0^n(x)|^2 dx \int |JX_t^\varepsilon|^2 |\varphi(X_t^\varepsilon)|^2 dx \\ &= \int |W_0^n(x)|^2 dx \int (JY_t^\varepsilon)^{-1} |\varphi(x)|^2 dx \end{aligned}$$

From lemma 2.1 we deduce

$$\mathbb{E} \left| \int W_0^n(x) (JY_t^\varepsilon)^{-1} \varphi(X_t^\varepsilon) dx \right|^2 \leq C \int |W_0^n(x)|^2 dx$$

Then passing to the limit as  $\varepsilon \rightarrow 0$  and  $n \rightarrow \infty$  in equation (3.28) we obtain

$$\lim_{n \rightarrow \infty} \int W(t, x) \varphi(x) dx = 0,$$

and this implies that

$$\lim_{n \rightarrow \infty} \int u^n(t, x) \varphi(x) dx = \int u(t, x) \varphi(x) dx,$$

□

### 3.5 Negative example

We considered the stochastic equation

$$\partial_t u(t, x) + \text{Div}\left((b(x - B_t) + \frac{dB_t}{dt}) \cdot u(t, x)\right) = 0,$$

which is equivalent to the deterministic continuity equation

$$\partial_t u(t, x) + \text{Div}(b(x)u(t, x)) = 0.$$

Then we do not expect to obtain the regularization effect by noise.

Now, we write the drift term in semimartingale form, applying the Ito formula to  $b(x - B_t)$  we have

$$b(x - B_t) = b(x) - \int_0^t b'(x - B_s)dB_s + \frac{1}{2} \int_0^t b''(x - B_s)ds.$$

Using the notation in our hypothesis we have  $b' = g$  and  $b'' = f$ . Thus we conclude that  $b$  satisfies our hypothesis only when it is regular.

### 3.6 A possible extension.

The main our tool in order to have estimations on the derivative of the flow was the Itô-Wentzell-Kunita formula. However, it is possible only to apply this formula for compositions of semimartingales. In order to generalize our result for more general  $b$  we have in mind to work in the context of the theory of stochastic calculus via regularization. This calculus was introduced by F. Russo and P. Vallois ( see [28] as general reference ) and it have been studied and developed by many authors. In the paper of F. Flandoli and F. Russo they obtain a Itô-Wentzell-Kunita formula for more general process, see [15] for details.

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